Computation and Analysis on Reinhardt Polygons with Multiple Prime Divisors

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Reinhardt Polygons

- $P$ is a convex polygon in the plane with $n$ sides
- $P$ has four relevant properties
  - Area
  - Perimeter
  - Diameter
  - Width
- There are 3 extremal problems
- Reinhardt polygons are optimal in all three problems
Reinhardt Polgons (cont.)

Polynomial: \( F(z) = 1 - z^{15} + z^{30} \)

Coefficient array: 

\[
[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
\]

Composition: [ 15 15 15 ]
Representations

1. Polygons
   - Must have a “star” cycle in its center (shown in blue)
   - Each angle in the star can be represented as a multiple of the angle $\frac{\pi}{n}$
   - All diameters cross one another
Representations (cont.)

2. Polynomials

- $F(z)$ is divisible by $\Phi_{2n}(z) = \Phi_n(-z)$
- $F(z)$ has alternating sign and all coefficients are elements of $\{1, 0, -1\}$
- $F(z)$ has an odd number of nonzero coefficient
- In most cases, we restrict $F(0) = 1$
Representations (cont.)

3. Compositions

- Composition of $n$ is a sequence of positive integers whose sum is $n$
- We represent a Reinhardt polynomial as a composition of the number of “gaps” between nonzero terms
  - Must always have an odd number of terms

For $n = 30$,

$$1 - z^4 + z^7 - z^8 + z^9 \Rightarrow [4 \ 3 \ 1 \ 1 \ 21]$$
Dihedral Equivalence

\[
\begin{bmatrix}
4 & 21 & 1 & 1 & 3 \\
3 & 4 & 21 & 1 & 1 \\
1 & 3 & 4 & 21 & 1 \\
1 & 1 & 3 & 4 & 21 \\
21 & 1 & 1 & 3 & 4
\end{bmatrix}
\quad \leftrightarrow 
\begin{bmatrix}
4 & 3 & 1 & 1 & 21 \\
3 & 1 & 1 & 21 & 4 \\
1 & 1 & 21 & 4 & 3 \\
1 & 21 & 4 & 3 & 1 \\
21 & 4 & 3 & 1 & 1
\end{bmatrix}
\]
Normalization

- Method to create one unique representation of a polygon

Start: [3 4 1 1 2 1 4 5 2 1 1 1 2 1 1]

[3 4 1 1 2 1 4 5 2 1 1 1 2 1 1]

End: [5 4 1 2 1 1 4 3 1 1 2 1 1 1 2]
Sporadic vs. Periodic

- **Periodic** Reinhardt polygons are those where the composition repeats periodically
  
  e.g. \([15 \ 15 \ 15] = [(15)^3]\)

- **Sporadic** polygons are Reinhardt polygons which are not periodic
  
  e.g. \([7 \ 4 \ 1 \ 1 \ 2 \ 3 \ 1 \ 2 \ 5 \ 5 \ 2 \ 1 \ 3 \ 2 \ 1 \ 1 \ 4]\)
Alternate Representation

• From previous work, we can define

\[ F(z) = f_1(z) \Phi_q(-z^{pr}) + f_2(z) \Phi_p(-z^{qr}) \]

\[ g_1(z) = f_1(z) \Phi_q(-z^{pr}) \]

e.g.

\[ f_1 \quad + 0 0 - + 0 \]

\[ g_1 \quad + 0 0 - + 0 - 0 0 + - 0 + 0 0 - + 0 \ldots \]

• This representation restricts the problem of finding valid Reinhardt polynomials (i.e. \( F(z) \)) to finding valid \( f_1 \) and \( f_2 \)
Previous Results

- The number of periodics is known for all $n$.
- For almost all $n \geq 105$, the number of sporadic polygons exceeds the number of periodic ones.
- Let $E_1(n)$ be the number of sporadic Reinhardt polygons for a given $n$.

$$ E_1(2pq) = \frac{2^{p-1}-1}{p} \cdot \frac{2^{q-1}-1}{q} $$
## Previous Data

<table>
<thead>
<tr>
<th>$n$</th>
<th>$E_1(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>3</td>
</tr>
<tr>
<td>45</td>
<td>144</td>
</tr>
<tr>
<td>60</td>
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<td>66</td>
<td>93</td>
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<td>78</td>
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<td>90</td>
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<tr>
<td>110</td>
<td>279</td>
</tr>
<tr>
<td>117</td>
<td>2,587,284</td>
</tr>
</tbody>
</table>
Second Construction

- Goal: find valid $f_1$ and $f_2$ such that they satisfy the alternate representation,

$$F(z) = f_1(z) \Phi_q(-z^{pr}) + f_2(z) \Phi_p(-z^{qr})$$

and the other necessary properties.
Second Construction

- We know \( F(z) = f_1(z) \Phi_q(-z^{pr}) + f_2(z) \Phi_p(-z^{qr}) \)
- Goal: find specific \( f_1 \) and \( f_2 \) such that this equation holds
- To do so, we utilize the following algorithm, also known as “the second construction”

Remember: \( p \) and \( q \) are distinct odd primes
\( r \geq 2, r \in \mathbb{Z} \)
\[ n = 30; \quad p = 3, \quad q = 5, \quad r = 2 \]

\[
A_1 = +0 \quad B_1 = 00 \quad A_2 = 0^- \quad B_2 = +^- \\
\]

\[
f_1 \quad g_1 + 00-+0-00+-0+00+-0-00+-0+00+-00+-00+-00+-0+00+-0 \\
\]

\[
g_2 000+-00++000+-00+-000+-00+-000+-00+-00+-00+-00+-00+-0 \\
\]

\[
F + 000000-++-+-0+-+++--+000+-++000+-++0000000 \\
\]

\[
[ 7 \quad 6 \quad 1 \quad 1 \quad 1 \quad 1 \quad 2 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 4 \quad 1 \quad 1 ] \\
\]
Second Construction (cont.)

- Let $c$ be a composition of $r$ into an even number of parts: $c = (r_1, r_2, \ldots, r_{2m})$
- We decompose $f_1$ and $f_2$ into blocks of $r$ terms, i.e.

$$f_1 = A_1 A_2 A_3 \ldots A_p \quad \text{degree}(f_1) < rp$$
$$f_2 = B_1 B_2 B_3 \ldots B_q \quad \text{degree}(f_2) < rq$$
Second Construction (cont.)

\[ A_i \]
\[
\begin{array}{cccccc}
  r_1 + 1 & r_2 - 1 & r_3 + 1 & r_4 - 1 & r_5 + 1 \\
  + \ldots + & 0 & - \ldots + & 0 & - \ldots + \\
\end{array}
\]

\[ B_j \]
\[
\begin{array}{cccccc}
  r_1 - 1 & r_2 + 1 & r_3 - 1 & r_4 + 1 & r_5 - 1 \\
  0 & - \ldots + & 0 & - \ldots + & 0 \\
\end{array}
\]
Second Construction (cont.)

\[ A_i \]

\[ r_1 + 1 \quad r_2 - 1 \quad r_3 + 1 \quad r_4 - 1 \quad r_5 + 1 \]

\[ + \ldots + \quad 0 \quad - \ldots + \quad 0 \quad - \ldots + \]

\[ \cdots \]

\[ B_j \]

\[ r_1 - 1 \quad r_2 + 1 \quad r_3 - 1 \quad r_4 + 1 \quad r_5 - 1 \]

\[ 0 \quad - \ldots + \quad 0 \quad - \ldots + \quad 0 \]

\[ \cdots \]

\[
A_{i,j} \in \begin{cases} 
S_o(r_1 + 1, (-1)^{i+1}s), & j = 1, \\
S_e(r_j + 1, (-1)^i s), & j \geq 3 \text{ odd}, \\
Z(r_j - 1), & j \text{ even}
\end{cases}
\]

\[
B_{i,j} \in \begin{cases} 
Z(r_j - 1), & j \text{ odd}, \\
S_e(r_j + 1, (-1)^i s), & j \text{ even}
\end{cases}
\]
$n = 30; \ p = 3, \ q = 5, \ r = 2$

$A_1 = 0+ \quad B_1 = 0- \quad A_2 = +0 \quad B_2 = 0-$

$C = +$

$f_1 \quad +-00000-+00000+00000-+00000+-00000$

$f_2 \quad 00++0+-00+-00+-00+-00+-00+$

$F \quad +-0+-+-+-00000-+-0000000+0000000-+$

\[76111112111111411\]
Analysis of Second Construction

• It has been proven that this algorithm only gives us Reinhardt polygons (although many dihedrals)
• Counting the number of periodics and sporadics we obtain a lower bound on $E_1(n)$ by dividing by the total number of possible dihedrals

$$E_1(n) \geq \frac{v}{r} \left( 2^r - 2 \cdot \frac{2^{r_o(p-1)} - 1}{p} \cdot \frac{2^{r_e(q-1)} - 1}{q} - \frac{U}{4pq} \right)$$
EXTENSIONS TO 3 DISTINCT PRIME DIVISORS
Third Construction

• We generalized the second construction to include a third distinct prime factor, \( l \), so now we have \( n = pqlr \) with \( r \geq 3 \)

• Goal is to determine nontrivial \( f_1(z), f_2(z), \) and \( f_3(z) \) such that

\[
F(z) = f_1(z)\Phi_q(-z^{lpr}) + f_2(z)\Phi_p(-z^{lqr}) + f_3(z)\Phi_l(-z^{pqr})
\]

is a valid Reinhardt polynomial
Third Construction (cont.)

- Let $c$ be a composition of $r$ into $m$ parts such that $m \equiv 0 \mod 3$: $c = (r_1, r_2, \ldots r_m)$
- We decompose $f_1$, $f_2$, and $f_3$ into blocks of $r$ terms, i.e.

\[
f_1 = A_1 A_2 A_3 \ldots A_{pq} \quad \text{degree}(f_1) < r_{pq}
\]
\[
f_2 = B_1 B_2 B_3 \ldots B_{pl} \quad \text{degree}(f_2) < r_{pl}
\]
\[
f_3 = C_1 C_2 C_3 \ldots C_{ql} \quad \text{degree}(f_3) < r_{ql}
\]
Third Construction (cont.)
Third Construction (cont.)

\[ A_{i,j} \in \begin{cases} 
S_o(r_j + 1, (-1)^i + 1) & \text{if } j = 1 \\
S_e(r_j + 1, (-1)^i) & \text{if } j = 1 \mod 3 \\
Z(r_j - 1) & \text{if } j = 0 \mod 3 \\
Z(r_j) & \text{if } j = 2 \mod 3 
\end{cases} \]

\[ B_{i,j} \in \begin{cases} 
S_e(r_j + 1, (-1)^i) & \text{if } j = 2 \mod 3 \\
Z(r_j - 1) & \text{if } j = 0 \mod 3 \\
Z(r_j) & \text{if } j = 1 \mod 3 
\end{cases} \]

\[ C_{i,j} \in \begin{cases} 
S_e(r_j + 1, (-1)^i) & \text{if } j = 0 \mod 3 \\
Z(r_j - 1) & \text{if } j = 1 \mod 3 \\
Z(r_j) & \text{if } j = 2 \mod 3 
\end{cases} \]
Computational Approach
Analysis of the Third Construction

- The total number of Reinhardt polygons produced by the third construction is
  \[ 2(pqr_1 + plr_2 + qlr_3) \]

- If \( r = 3 \), i.e. \( r_1 = r_2 = r_3 = 1 \), then the number of periodics produced is

\[
2(ql+q+l+p-1) + 2(pl+p+l+q-1) + 2pq+p+q+l-1 - 2(2l+p+q-1) - 2(2q+l+p-1) - 2(2p+p+l-1)
\]
\[ E_1(pqlr) \geq \frac{v}{4pqlr} \left( 2(pqr_1 + plr_2 + qlr_3) \right. \]
\[ \left. - 2(qlr_3 + qr_1 + lr_2 + (p-1)\max(r_1, r_2)) \right. \]
\[ \left. - 2(plr_2 + pr_1 + lr_2 + (q-1)\max(r_1, r_3)) \right. \]
\[ \left. - 2(pqr_1 + pr_2 + qr_3 + (l-1)\max(r_2, r_3)) \right. \]
\[ \left. + 2(l(r_2 + r_3) + \min((p+q-1)r_1, pr_1 + (q-1)r_4, qr_1 + (p-1)r_2)) \right. \]
\[ \left. + 2(q(r_1 + r_3) + \min((l+p-1)r_2, lr_2 + (p-1)r_1, pr_2 + (l-1)r_3)) \right. \]
\[ \left. + 2(p(r_1 + r_2) + \min((q+l-1)r_3, qr_3 + (l-1)r_2, lr_3 + (q-1)r_1)) \right) \]
Analysis (cont.)

- If we let $r = 3$, i.e. $r_1 = r_2 = r_3 = 1$ in the previous equation, and $n = pqlr$, we obtain

$$E_1(n) \geq \frac{\nu}{4pqlr} \left( 2^{(p+1+l-1)} \right).$$

$$\left[ 2^{(pq+lp+qp-p-q-l+1)} - 2^{ql} - 2^{pq} - 2^{pl} + 2^l + 2^q + 2^p \right]$$
Comparisons

- For the second construction, the lower bound gives us approximately
  \[ E_1(315) \geq 1.13 \cdot 10^{16} \]

- The third construction gives us approximately
  \[ E_1(315) \geq 5.62 \cdot 10^{18} \]
What’s Next?

• Can we combine different compositions and obtain a better lower bound?

• Can we find a formula for $E_1(3pql)$?

• Does the second/third construction give us all Reinhardt polygons?
FINDING SPORADIC 105-GONS
Relevant Challenges

1. Computational Limits
   - Current solution: restricting search space

2. Determining Uniqueness
   - Current solution: normalization
Previously Known Results

- There are approximately 245 million periodic Reinhardt 105-gons
- We believe there are over 350 million sporadic 105-gons
- We can only determine around 60% of all sporadic Reinhardt 105-gons using the second construction
New Generation Methods

- Enumerating all possible $f_1$, $f_2$, and $f_3$ is computationally impossible
  - This is on order of $O(2^k)$

- Requires a new approach: if we can restrict one of $f_1$, $f_2$, or $f_3$, then we can obtain a more attainable runtime
First Program

• For $n = 105$, $f_1$ has degree 35, $f_2$ has degree 21, and $f_3$ has degree 15

• We can cycle through all possibilities for $f_1$ and $f_2$ and then restrict the possible $f_3$’s

  e.g. 
  $f_1 \ldots + - \ldots$
  $f_2 \ldots + - \ldots$
  $f_3 \ldots - + \ldots$

  $\overline{F} \ldots + - \ldots$
Not Enough Restrictions

• The previous technique is still exponential
  – But we can specify unique base cases and obtain some results

• How else can we generate more sporadic 105-gons?
Adding Polynomials

\[ F(z) = F'(z) + F''(z) - F'''(z) \]

- Maintains divisibility by \( \Phi_{2n}(z) \) implicitly
- Check for alternating, an odd number of nonzero terms, and \( F(0) = 1 \)
Applications

We applied addition to two problems:

1. Determining a generating set for $n = 45$
   - Helps test the validity of the method
     - All $n = 45$ sporadic polygons are known

2. Finding previously unknown 105-gons
Generating Sets for $n = 45$

- We found a generating set
- Example: 20 sporadic polygons, the $G_i$ which generate the entire sporadic set, $S$

Let $F \in S$, we obtain $F = G_1 - G_3 + G_9 - G_{15} + G_{17}$
Adding to Obtain More 105-gons

- Start by using the first program to generate a small set of 105-gons

- Add the 105-gons together, combine the results and then iterate
New Results for $n = 105$

- We’ve been able to generate over 135,823 sporadic Reinhardt polygons
- 87,773 had not been previously determined by the second construction
What’s Next?

• Maximizing computational power and obtaining adequate space management to find all sporadic Reinhardt 105-gons

• Can we find a generating set for $n \geq 45$?

• What properties define elements in the generating set from others?
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Questions?